Stability Analysis for Semiconductor Lasers with Optical Feedback.

José Oliverio Alvarez Sierra
Progam in Applied of Mathematics, University of Arizona
Tucson, AZ85721; e-mail:joas@math.arizona.edu

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Abstract

The behavior of a semiconductor laser with and without optical feedback is modeled by the Lang-Kobayashi rate equations. I present the stability analysis for both cases and describe the dynamics of a semiconductor laser when increasing the feedback level, in particular the phenomenon called low-frequency fluctuations or Sisyphus effect.

1 Introduction

Semiconductor Lasers, in any typical application, are subject to optical feedback. The feedback, often time delayed, generally comes from the end mirrors of the laser cavity or from any optical component in the system. Therefore, the dynamical behavior of a semiconductor laser subject to optical feedback is given by differential equations with time delay.

One interesting phenomenon that occurs with semiconductor lasers at moderate levels of feedback strength is called *low-frequency fluctuations* (LLF's) in the power output of the laser. This phenomenon consists of power drops, even down to zero at different intervals in time, followed by gradual recovery. Beacuse of this pattern this phenomenon is also called *Sisyphus Effect*, this behavior can be transient, i.e. the output can become stable after a period of initial transient time, or it can be sustained.

Experiments have shown that the behavior of a semiconductor laser diode with optical feedback has five regimes (I-V) of qualitative different effects, according to the feedback level[1]. For very weak feedback levels, regime I, the laser operates at a single external cavity mode (ECM) which is close to the laser frequency without the feedback, i.e. solitary laser frequency. In regime II, when feedback level is increased, the laser shows noise-induced hopping between several external cavity modes. The rate of mode hopping decreases for increasing feedback level. In regime III, the laser operates only in one dominant mode, which is the mode with minimum linewidth. When passing from regime III to IV, by increasing the feedback level, bifurcations appear during this transition and once we get to regime IV coherence colapse and low-frequency fluctuations occur. Above in regime V, we have a single-mode narrow linewidth operation, but in order to reach this regime of very strong feedback it is necessary to antireflection coat the laser facet facing the external cavity or to use a very low bias current [4]. I will mainly focus on the transition from regime I to regime IV.

Describing a little more in detail we have [2]: For weak feedback levels (regimes I-II) the laser at a single ECM, when the feedback level is incresed (regime III) the ECM becomes unstable and is replaced by a limit cycle due to a Hopf bifurcation. Then (regime IV) the system follows a period-doubling or a quasiperiodic route to chaos[4]. A new chaotic attractor is created, that creates a stable fixed point(called *mode*) and remains in the vicinity of the original ECM. At the same time a new ECM with higher frequency and higher gain is born via a saddle-node bifurcation that creates an unstable saddle fixed point, called *antimode*. These two attractors initially exist independently, each having its own basin of attraction, while a stable manifold of the antimode is the basin boundary. As we increase the feedback level, but remain still in regime IV, the chaotic attractor of the first ECM starts crossing into the basin of the second ECM, simultaneously, the second ECM goes through similar process than the first one, starting with a Hopf bifurcation and resulting in a chaotic attractor that grows in size with increasing feedback level and an antimode. The third attractor (ECM) is created from this Hopf Bifurcation. At some point the second attractor undergoes either a merging crisis with the ruins of the first attractor (orginal ECM) or experiences a boundary crisis into the third ECM, which is now accesible to the system and has the highest gain. The above pattern repeats for increasing feedback level. The LFF happen usually at moderate feedback levels when the system visits many attractor ruins corresponding to differrent ECMs in a specific pattern.

In the LFF regime the system moves among attractors in the direction of the highest gain and then goes back to the low gain modes in the process governed bythe antimodes, this is in detail the Sisyphus effect. The highest gain ECM can be stable, if the basin of this ECM, and the other attractor ruins are separated then the LFF are sustained, since the highest gain mode is not accesible to the system. If the basin of the highest gain mode is accesible to the system then LFF are transient, because a trajectory can enter its vicinity and stay there.

It is known that nonlinear gain leads to the stabilization of ECMs, so that the onset of chaos and chaotic transitions are shifted towards higher feedback levels, but the number and frequencies of ECMs at a given feedback level stay the same[5]. Therefore when the gain saturation coefficient ε is different from zero, all ECMs can become stable (or quasi-stable) and we will not have chaotic behavior. For this reason in this project I will only consider the case $\varepsilon = 0$, the most interesting case.

When investigating optical feedback effects, researchers based their analysis on the *Lang-Kobayashi* rate equations. They describe a laser with optical feedback from an external mirror reflector. In this project I will focus on the case of a flat vertical external mirror.

For this project I did all the stability analysis for the case of a semiconductor laser without feedback(sec.3.1). For the case with the feedback I found the stationary solutions, analyzed their behavior (with exception of equation (18)) and from equations (28) to (33) I reproduced results.

2 Lang-Kobayashi Equations

The dynamics of semiconductor lasers subject to optical feedback is described by the Lang-Kobayashi equations for the intracavity complex field $E(t)$ and the carrier population $N(t)$, which are given by:

$$\frac{dE(t)}{dt} = \frac{1+i\alpha}{2}(G - \tau_p^{-1}E(t)) + \gamma e^{-i\omega_o\tau}E(t-\tau), \tag{1}$$

$$\frac{dN(t)}{dt} = J - \frac{N(t)}{\tau_s} - G|E(t)|^2, \tag{2}$$

where α is the linewidth enhancement factor, ω_0 is the laser frequency without the feedback, τ_p is the photon lifetime, τ_s is the carrier lifetime and J is the pumping term. The external cavity parameters are the feedback parameter $\gamma = \frac{\kappa}{\tau_{in}}$, where τ_{in} is the round trip time in the laser cavity and κ^2 is the power reflected from the external cavity. Since we are going to work with weak and moderate feedback we must have that $\kappa \ll 1$ [4] . Also the delayed time $\tau = \frac{2L_{ext}}{c}$, which is the round-trip time of the light in the external cavity of lenght L_{ext}. The modal gain per unit time is $G = G_N(N - N_0)(1 - \varepsilon E^2)$, where G_N is the gain constant and N_0 is the carrier density at transparency. The electric field is normalized so that $V_c|E(t)|^2$ is the total photon number in the laser wave guide. The parameter $J_{th} = \frac{N_{th}}{\tau_s}$ is the lasing threshold current density of a solitary laser and N_{th} is the threshold carrier density.

We can normalize the Lang-Kobayashi equations by measuring time in units of the photon lifetime ($\frac{t}{\tau_p} \to t$) and introduce the normalized excess carrier density $n(t) = \frac{N(t)-N_{th}}{N_{th}}$, thus the Lang-Kobayashi equations are rewritten as follows:

$$\frac{dE(t)}{dt} = \frac{1+i\alpha}{2}gn(t)E(t)) + \eta e^{-i\omega_o\tau}E(t-\tau), \tag{3}$$

$$T\frac{dn(t)}{dt} = P - n(t) - (gn(t)+1)|E(t)|^2, \tag{4}$$

where $g = \tau_p G_N N_{th}$, $\eta = \tau_p\gamma$, $T = \frac{\tau_s}{\tau_p}$ and$P = \frac{(J-J_{th})}{J_{th}}$. It is important to notice that the above equations describe the dynamics of three independent variables, since $E(t) = E_{real} + iE_{imag}$.

3 Stability Analysis

3.1 Stability Without Feedback

We first consider the Lang-Kobayashi system without delay, i.e. without feedback. In order to do this we add the equation for E^* to the system,we thus get

$$\frac{dE(t)}{dt} = \frac{1+i\alpha}{2}gn(t)E(t)), \tag{5}$$

$$\frac{dE^*(t)}{dt} = \frac{1-i\alpha}{2}gn(t)E^*(t)), \tag{6}$$

$$T\frac{dN(t)}{dt} = P - n(t) - (gn(t)+1)|E(t)|^2. \tag{7}$$

In this system we have two fixed points $|E(t)|^2 = 0$, $n(t) = P$ and $E(t) = \sqrt{P}$, $E^*(t) = \sqrt{P}$, $n(t) = 0$.

In order to find the eigenvalues for $|E(t)|^2 = 0$ and $n(t) = P$, we compute

$$\mathbf{J_1} \equiv \begin{pmatrix} gP\frac{1+i\alpha}{2} & 0 & 0 \\ 0 & gP\frac{1-i\alpha}{2} & 0 \\ 0 & 0 & \frac{-1}{T} \end{pmatrix},$$

Clearly we have $\lambda_1 = gP\frac{1+i\alpha}{2}$,$\lambda_2 = gP\frac{1-i\alpha}{2}$ and $\lambda_3 = \frac{-1}{T}$. So if P is positive the fixed point is unstable and if is negative the fixed point is stable.

The eigenvalues for $E(t) = \sqrt{P}$, $E^*(t) = \sqrt{P}$ and $n(t) = 0$, are found by computing

$$\mathbf{J_2} \equiv \begin{pmatrix} 0 & 0 & g\sqrt{P}\frac{1+i\alpha}{2} \\ 0 & 0 & g\sqrt{P}\frac{1-i\alpha}{2} \\ -\frac{\sqrt{P}}{T} & -\frac{\sqrt{P}}{T} & \frac{-1-gP}{T} \end{pmatrix},$$

for which we obtain the following characteristic polynomial:

$$-\lambda\left[\lambda^2 + \lambda\left(\frac{1+gP}{T}\right) + \frac{gP}{T}\right] = 0, \tag{8}$$

so that $\lambda_1 = 0$ and $\lambda_{2,3} = -\frac{1+gP}{2T} \pm \frac{1}{2T}\sqrt{(1+gP)^2 - 4gPT}$. Working with the discriminant we obtain

$$P = \frac{(2T-1)}{g} \pm \frac{2}{g}\sqrt{T^2 - T}$$

Now taking into account that T and g are positive (see table at the end), P ranges over $(-1, \infty)$ and the fact that we obtain two positive values for P, $0 < P_1 \ll 1$ and $P_2 \gg 1$ we see that the system is stable spiral for P values in (P_1, P_2) and stable node for values in $(-1, P_1)$ or (P_2, ∞), experimentally we do not see a stable node in any interval $(0, h)$ for any small positive h, this is due to the noise in the experiment, and for experiments that use values of P as big as P_2 the Lang-Kobayashi equations do not contain all dominant effects.

There are other methods that have been applied to find the stability for the system without feedback and with weak feedback levels (Regimes I-III). For example, [3] uses the method of principle of the argument, a powerful tool from the theory of complex functions, but the approximations used there are only valid for weak feedback levels, before the first pair of fixed points is created.

3.2 Stability With Feedback

We now consider the Lang-Kobayashi equations with the feedback $(3-4)$. To perform the stability analysis we write the complex electric field as

$$E(t) = E_0(t)e^{i(\phi(t)-\omega_0 t)} \tag{9}$$

and we look for stationary solutions of the form

$$E(t) = E_0(t)e^{i(\omega_s-\omega_0)t} \text{ and } n(t) = n_0$$

plugging them into the equations (3), (4) and separating equation (3) in real and imaginary parts we get the following equations

$$\frac{gn_0}{2} + \eta\cos(\tau\omega_s) = 0, \tag{10}$$

$$\frac{\alpha gn_0}{2} - \eta\sin(\tau\omega_s) = (\omega_s - \omega_0), \tag{11}$$

$$\frac{1}{T}(P - n_0(gn_0+1)E_0^2) = 0. \tag{12}$$

Substituting $\frac{gn_0}{2}$ from (10) into (11), we get:

$$-\alpha\eta\cos(\tau\omega_s) - \eta\sin(\tau\omega_s) = (\omega_s - \omega_0). \tag{13}$$

Solving this equation for ω_0 we obtain:

$$\omega_0 = \omega_s + \eta\left(\alpha\cos(\omega_s\tau) + \sin(\tau\omega_s)\right), \tag{14}$$

which is equivalent to

$$\omega_0 = \omega_s + \eta\sqrt{\alpha^2+1}\sin(\omega_s\tau + \arctan\alpha). \tag{15}$$

Solving for E_0 in (12) we obtain:

$$E_0^2 = \frac{P - n_0}{gn_0 + 1}. \tag{16}$$

and since the above equations gave us ω_s and E_0 we solve for n_0 in (10) and get

$$n_0 = -\eta\cos(\omega_s\tau) \tag{17}$$

Now, at low feedback levels, equation (15) has only one solution, which is, as expected, close to the solitary laser frequency ω_0. If we increase the feedback level, i.e. η, additional solutions

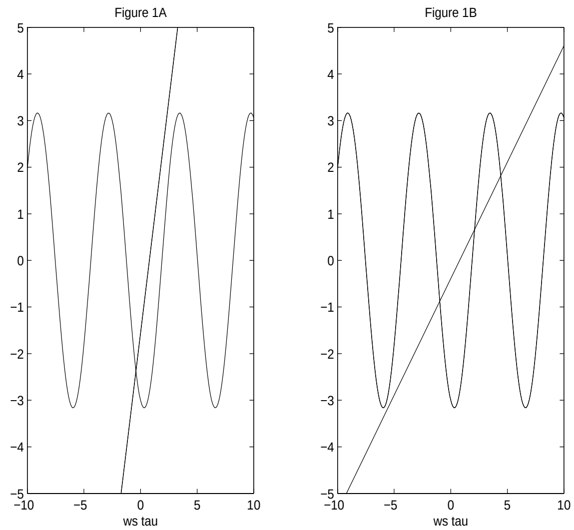

Figure 1: Ploting $\frac{1}{\eta}(\omega_s\tau - \omega_0\tau)$ and $-\sqrt{1+\alpha^2}\sin(\omega_s\tau + \arctan\alpha)$, with $\alpha = 3$ and $\omega_0\tau = \frac{\pi}{4}$. In Figure 1A we used $\eta = 0.5$ and in Figure 1B, $\eta = 2$. Note that the solutions are created in pairs.

appear in pairs (see Figure 1). Since we have the condition $E_0 \geq 0$, some solutions of (15) are not allowed.

The feedback level at which a new pair of fixed-point solutions is created can be approximated by the formula [2]

$$\eta \approx \frac{\beta + \sqrt{\beta^2 - 2}}{2\tau\sqrt{\alpha^2+1}}. \tag{18}$$

where $\beta = \omega_0\tau + \arctan\alpha + \left(2n - \frac{1}{2}\right)\pi$, and $n = 1, 2, \ldots.$ The approximation improves with increasing η.

Now, to analize the stability of the stationary solutions we found, we need to recall that since $E(t) = E_0(t)e^{i(\phi-\omega_0 t)}$, we can cast the Lang-Kobayashi equations (1), (2) as:

$$\frac{dE_0(t)}{dt} = \frac{1}{2}G_N\Delta N(t)E_0(t) + \eta E_0(t-\tau)\cos(\Delta(t)) + \frac{R}{2VE_0(t)}, \tag{19}$$

$$\frac{d\phi(t)}{dt} = \frac{1}{2}\alpha G_N\Delta N(t) - \eta\frac{E_0(t-\tau)}{E_0(t)}\sin\Delta(t), \tag{20}$$

$$\frac{dN(t)}{dt} = J - \frac{N(t)}{\tau_s} - G|E(t)|^2, \tag{21}$$

where $\Delta(t) = \omega_0\tau + \phi(t) - \phi(t-\tau)$ and R is the rate of spontaneouos emision into the lasing mode, $\varepsilon = 0$ and the rest of the parameters have already been defined.

For $\kappa \ll 1$, the effective reflectivity of the laser mirror facing the external cavity is [6]

$$r(\omega) \simeq r_2(1 + \kappa e^{-i\omega\tau}) \tag{22}$$

where r_2 is the amplitude reflectivity of the laser mirror, hence the excess reflectivity is given by

$$\Delta r(\omega) = |r(\omega)| - r_2 \simeq r_2 \kappa \cos(\omega\tau). \tag{23}$$

By (17) we have

$$n_0 = -\frac{\Delta r}{\tau_{in} r_2} \tag{24}$$

The dominant mode will therefore be the mode with maximum excess reflectivity, also it has an absolute maximum for $\omega\tau = 0 \pmod{2\pi}$. Indeed we can see this by multiplying both sides of (15) by τ, giving us

$$\omega_0 \tau = \omega_s \tau + \eta\tau\sqrt{\alpha^2+1}\sin(\omega_s\tau + \arctan\alpha) \tag{25}$$

in which case we have $\omega_0\tau = \alpha\tau\eta \pmod{2\pi}$, which is easily obtained by the fact that $\sqrt{1+\alpha^2} = \sec(\arctan\alpha)$. We thus define a detuning parameter by

$$d = \omega_0\tau - \alpha\tau\eta (mod 2\pi) \tag{26}$$

with this we have that when $d = 0 \pmod{2\pi}$, we have a maximum for Δr. Now, the limit where the frequency jumps to another external cavity mode, because the old ECM is no longer a solution of (15) it corresponds to a maximum in ω_0 as a function of ω and using (15) is given by

$$\frac{d\omega_0}{d\omega} = 1 + \eta\tau\sqrt{1+\alpha^2}\cos(\omega\tau + \arctan\alpha) = 0 \tag{27}$$

Stability can be determined by seeking deviations of the stationary solutions of (15) that are proportional to e^{zt}. Doing this, we obtain the system determinant:

$$D(z) = \left(z + \frac{1}{\tau_R}\right)\left[(z+f_c)^2 + f_z^2\right] + \omega_R^2\left[z + f_c + \alpha f_z\right] \tag{28}$$

where

$$f_c = \eta(1 - e^{-z\tau})\cos(\omega\tau) \tag{29}$$
$$f_z = \eta(1 - e^{-z\tau})\sin(\omega\tau) \tag{30}$$
$$\frac{1}{\tau_R} = \frac{1}{\tau_s} + G_N(E_0^2) \tag{31}$$

and

$$\omega_R^2 = G_N G(N_0)(E_0)^2 \tag{32}$$

The dynamic stability of a solution of (15) is determined by the position of the zeroes of the system determinant (28). A solution will be unstable if (28) has a zero in the right half z plane. Following the motion of the zeroes of $D(z)$ varying ω, we can determine the stability intervals for ω and consequently for the detuning parameter d [6].

As $\omega\tau$ passes a maximum of $\omega_0\tau$, i.e. where the frequency jumps to another ECM, when $\omega\tau$ is increasing, a zero of $D(z)$ moves in to the right half of the z plane along the real axis and it returns when $\omega\tau$ passes the minimum of $\omega_0\tau$. Therefore our condition for stability is $\frac{d\omega_0}{d\omega} > 0$, which gives us:

$$\frac{d\omega_0}{d\omega} = 1 + \eta\tau\sqrt{1+\alpha^2}\cos\left(\omega\tau + \arctan\alpha\right) > 0 \tag{33}$$

which is the stability condition I wanted to get.

From (15) and (33) we can see that from the two new fixed points that are created by increasing the feedback level, one is stable (mode), in particular is a chaotic attractor, and the other one is an unstable fixed point of the saddle type (antimode). The dynamical union of all these chaotic attractors is called the *Sisyphus attractor*, and is here where the Sisyphus effect takes place. This Attractor coexists with the highest gain mode, which is another chaotic attractor. If this last attractor is accesible to the system then the LFFs are transient. If it is not accesible then the LFFs are sustained.

I now give a table of values for the parameters of the system.

Parameter	*Symbol*	*Value*
Linewidth enhacement factor	α	3-7
Photon Lifetime	τ_p	2 ps
Carrier Lifetime	τ_s	2 ns
Laser Cav. Round Trip	τ_{in}	8 ps
Modal gain Coefficient	G_N	$1 \times 10^{-12}\ m^3 s^{-1}$
Carrier Dens. at Transp.	N_0	$1 \times 10^{24}\ m^{-3}$
Threshold current	J_{th}	$2 \times 10^{33}\ m^{-3} s^{-1}$
Injected current	J	$1 - 2J_{th}$
Volume of active region	V_c	$4 \times 10^{-16}\ m^3$
Feedback Level	γ	$0 - 30 \times 10^{-9}\ s^{-1}$
Delay time	τ	$1 - 10$ ns
Feedback Phase	$\omega_0\tau$	$-\pi - \pi$ rad

4 Conclusion

The problem was to determine the stationary solutions of the Lang-Kobayashi equations and their stability properties.

For the case without feedback I got a transcritical bifurcation with a bifurcation value of $0 \leq P \ll 1$. The branch $|E|^2 = 0$ of the bifurcation was unstable and the branch $|E|^2 = P$ stable. Here system works at solitary laser frequency.

In the case with the feedback I showed that for very weak feedback the system works at a single ECM, which is closed to the solitary laser frequency. Then, by increasing the feedback level, fixed points are created in pairs, one of these two is a chaotic attractor and the other is an unstable saddle fixed point.

There are several ways one can go from here, one of them is analyzing in more detail the LLFs to determine when are they transient or sustained, and if it is sustained, look for ways to apply control to induce escape of a trajectory from the Sisyphus attractor to the stable attractor so that to eliminate the power-dropout events.

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